

Chapter 8. Introduction to Trigonometry

Question-1

Simplify the following expressions: $\frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta}$.

Solution:

$$\frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta} = \frac{(\sin \theta + \cos \theta)(\sin^2 \theta - \sin \theta \cos \theta + \cos^2 \theta)}{\sin \theta + \cos \theta} = 1 - \sin \theta \cos \theta.$$

Question-2

Prove $\frac{\tan \theta}{1 - \cot \theta} + \frac{\cot \theta}{1 - \tan \theta} = 1 + \tan \theta + \cot \theta$.

Solution:

$$\text{L.H.S} = \frac{\tan \theta}{1 - \cot \theta} + \frac{\cot \theta}{1 - \tan \theta} = \frac{-\tan^2 \theta}{1 - \tan \theta} + \frac{\cot \theta}{1 - \tan \theta} = \frac{-\tan^2 \theta + \cot \theta}{1 - \tan \theta}$$

Multiply $\tan \theta / \tan \theta$ we get,

$$\begin{aligned} &= \frac{1 - \tan^3 \theta}{\tan \theta (1 - \tan \theta)} \\ &= \frac{(1 - \tan \theta)(1 + \tan \theta + \tan^2 \theta)}{\tan \theta (1 - \tan \theta)} \\ &= \frac{(1 + \tan \theta + \tan^2 \theta)}{\tan \theta} \\ &= \cot \theta + 1 + \tan \theta = \text{R.H.S.} \end{aligned}$$

Question-3

Prove $\tan^2 \phi + \cot^2 \phi + 2 = \sec^2 \phi \operatorname{cosec}^2 \phi$.

Solution:

$$\text{L.H.S} = \tan^2 \phi + \cot^2 \phi + 2$$

$$= \sec^2 \phi - 1 + \operatorname{cosec}^2 \phi - 1 + 2$$

[Using Identity $1 + \cot^2 \phi = \operatorname{cosec}^2 \phi$ and $1 + \tan^2 \phi = \sec^2 \phi$]

$$= \sec^2 \phi + \operatorname{cosec}^2 \phi = \frac{1}{\cos^2 \phi} + \frac{1}{\sin^2 \phi}$$

$$= \frac{\sin^2 \phi + \cos^2 \phi}{\cos^2 \phi \sin^2 \phi}$$

$$= \frac{1}{\cos^2 \phi \sin^2 \phi} \quad [\text{Using Identity } \sin^2 \phi + \cos^2 \phi = 1]$$

$$= \sec^2 \phi \operatorname{cosec}^2 \phi = \text{R.H.S.}$$



Question-4

Evaluate: $\left(\frac{\sin 27^\circ}{\cos 63^\circ}\right)^2 + \left(\frac{\cos 63^\circ}{\sin 27^\circ}\right)^2$.

Solution:

$$\begin{aligned}\left(\frac{\sin 27^\circ}{\cos 63^\circ}\right)^2 + \left(\frac{\cos 63^\circ}{\sin 27^\circ}\right)^2 &= \left(\frac{\sin(90^\circ - 63^\circ)}{\cos 63^\circ}\right)^2 + \left(\frac{\cos(90^\circ - 27^\circ)}{\sin 27^\circ}\right)^2 \\&= \left(\frac{\cos 63^\circ}{\cos 63^\circ}\right)^2 + \left(\frac{\sin 27^\circ}{\sin 27^\circ}\right)^2 \\&= 1 + 1 \\&= 2.\end{aligned}$$

Question-5

Prove that: $\frac{\cos(90^\circ - \theta)}{\sin \theta} + \frac{\sin \theta}{\cos(90^\circ - \theta)} = 2, \theta \neq 0^\circ$.

Solution:

$$\begin{aligned}\text{L.H.S} &= \frac{\cos(90^\circ - \theta)}{\sin \theta} + \frac{\sin \theta}{\cos(90^\circ - \theta)} \\&= \frac{\sin \theta}{\sin \theta} + \frac{\sin \theta}{\sin \theta} \\&= 1 + 1 \\&= 2 \\&= \text{R.H.S.}\end{aligned}$$

Question-6

Prove that: $\sec^2 \theta - \cot^2 (90^\circ - \theta) = \cos^2 (90^\circ - \theta) + \cos^2 \theta$.

Solution:

$$\begin{aligned}\text{L.H.S} &= \sec^2 \theta - \cot^2 (90^\circ - \theta) \\&= \sec^2 \theta - \tan^2 \theta \\&= \frac{1}{\cos^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} \\&= \frac{(1 - \sin^2 \theta)}{\cos^2 \theta} \\&= \frac{\cos^2 \theta}{\cos^2 \theta} = 1 \\ \text{R.H.S} &= \cos^2 (90^\circ - \theta) + \cos^2 \theta \\&= \sin^2 \theta + \cos^2 \theta \\&= 1 \\ \text{L.H.S} &= \text{R.H.S}\end{aligned}$$

Question-7

Prove that: $\frac{\cos(90^\circ - \theta) \cos \theta}{\tan \theta} + \cos^2(90^\circ - \theta) = 1$.

Solution:

$$\begin{aligned}\text{L.H.S} &= \frac{\cos(90^\circ - \theta) \cos \theta}{\tan \theta} + \cos^2(90^\circ - \theta) \\&= \frac{\sin \theta \cos \theta}{\tan \theta} + \sin^2 \theta \\&= \cos^2 \theta + \sin^2 \theta \quad [\text{Using Identity } \sin^2 \theta + \cos^2 \theta = 1] \\&= 1 \\&= \text{R.H.S.}\end{aligned}$$

Question-8

Prove that: $\cos(81^\circ + \theta) = \sin(9^\circ - \theta)$.

Solution:

$$\begin{aligned}\text{L.H.S} &= \cos(81^\circ + \theta) \\&= \cos[90^\circ - (9^\circ - \theta)] \\&= \sin(9^\circ - \theta) \\&= \text{R.H.S.}\end{aligned}$$

Question-9

Prove that: $\sin \theta \cos(90^\circ - \theta) + \cos \theta \sin(90^\circ - \theta) = 1$.

Solution:

$$\begin{aligned}\text{L.H.S} &= \sin \theta \cos(90^\circ - \theta) + \cos \theta \sin(90^\circ - \theta) \\&= \sin \theta \sin \theta + \cos \theta \cos \theta = \sin^2 \theta + \cos^2 \theta \quad [\text{Using Identity } \sin^2 \theta + \cos^2 \theta = 1] \\&= 1 \\&= \text{R.H.S.}\end{aligned}$$

Question-10

If $\frac{2 \sin \theta}{1 + \cos \theta + \sin \theta} = y$, then $\frac{1 - \cos \theta + \sin \theta}{1 + \sin \theta}$ is also y .

Solution:

$$\frac{1 - \cos \theta + \sin \theta}{1 + \sin \theta} = \frac{1 - \cos \theta + \sin \theta}{1 + \sin \theta} \times \frac{1 + \cos \theta + \sin \theta}{1 + \cos \theta + \sin \theta}$$



$$\begin{aligned}
 &= \frac{(1 + \sin \theta)^2 - \cos^2 \theta}{(1 + \sin \theta)(1 + \cos \theta + \sin \theta)} \\
 &= \frac{(1 + \sin \theta)(1 + \sin \theta) - [(1 + \sin \theta)(1 - \sin \theta)]}{(1 + \sin \theta)(1 + \cos \theta + \sin \theta)} \\
 &= \frac{(1 + \sin \theta)[(1 + \sin \theta) - (1 - \sin \theta)]}{(1 + \sin \theta)(1 + \cos \theta + \sin \theta)} \\
 &= \frac{2 \sin \theta}{1 + \cos \theta + \sin \theta} \\
 &= y \quad \left(\text{Since } \frac{2 \sin \theta}{1 + \cos \theta + \sin \theta} = y \right).
 \end{aligned}$$

Question-11

Prove: $\frac{1 - \tan^2 \phi}{\cot^2 \phi - 1} = \tan^2 \phi$.

Solution:

$$\begin{aligned}
 \text{L.H.S} &= \frac{1 - \tan^2 \phi}{\cot^2 \phi - 1} \\
 &= \frac{1 - \frac{\sin^2 \phi}{\cos^2 \phi}}{\frac{\cos^2 \phi}{\sin^2 \phi} - 1} \\
 &= \frac{\frac{\cos^2 \phi - \sin^2 \phi}{\cos^2 \phi}}{\frac{\cos^2 \phi - \sin^2 \phi}{\sin^2 \phi}} \\
 &= \frac{\sin^2 \phi}{\cos^2 \phi} \\
 &= \tan^2 \phi \\
 &= \text{RHS.}
 \end{aligned}$$

Question-12

Prove: $(\sec \theta - \tan \theta)^2 = \frac{1 - \sin \theta}{1 + \sin \theta}$.

Solution:

$$\begin{aligned}
 \text{LHS} &= (\sec \theta - \tan \theta)^2 \\
 &= \left(\frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta} \right)^2 \\
 &= \frac{(1 - \sin \theta)(1 - \sin \theta)}{(1 - \sin^2 \theta)} \\
 &= \frac{(1 - \sin \theta)(1 - \sin \theta)}{(1 - \sin \theta)(1 + \sin \theta)} \\
 &= \frac{1 - \sin \theta}{1 + \sin \theta} \\
 &= \text{RHS.}
 \end{aligned}$$

Question-13

Prove that $\frac{(1 + \cot A + \tan A)(\sin A - \cos A)}{\sec^3 A - \operatorname{cosec}^3 A} = \sin^2 A \cos^2 A$.

Solution:

$$\begin{aligned}\text{L.H.S} &= \frac{(1 + \cot A + \tan A)(\sin A - \cos A)}{\sec^3 A - \operatorname{cosec}^3 A} \\&= \frac{\left(1 + \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}\right)(\sin A - \cos A)}{\frac{1}{\cos^3 A} - \frac{1}{\sin^3 A}} \\&= \frac{(\cos^2 A + \sin^2 A + \cos A \sin A)(\sin A - \cos A)}{\frac{\sin A \cdot \cos A}{\sin^3 A - \cos^3 A}} \\&= \frac{\sin^3 A - \cos^3 A}{\cos^3 A \sin^3 A} \\&= \sin^2 A \cos^2 A \\&= \text{R.H.S.}\end{aligned}$$

Question-14

If $a \cos \theta - b \sin \theta = c$, show that $a \sin \theta + b \cos \theta = \pm \sqrt{a^2 + b^2 - c^2}$.

Solution:

$$\begin{aligned}(a \cos \theta - b \sin \theta)^2 + (a \sin \theta + b \cos \theta)^2 &= a^2(\cos^2 \theta + \sin^2 \theta) + b^2(\sin^2 \theta + \cos^2 \theta) - 2ab \sin \theta \cos \theta + 2ab \sin \theta \cos \theta \\&= a^2 + b^2 \therefore (a \cos \theta - b \sin \theta)^2 + (a \sin \theta + b \cos \theta)^2 = a^2 + b^2 \\ \therefore (a \cos \theta - b \sin \theta)^2 &= a^2 + b^2 - (a \sin \theta + b \cos \theta)^2 \\&= a^2 + b^2 - c^2 \therefore a \sin \theta + b \cos \theta = \pm \sqrt{a^2 + b^2 - c^2}.\end{aligned}$$

Question-15

Prove that $2(\sin^6 \theta + \cos^6 \theta) - 3(\sin^4 \theta + \cos^4 \theta) + 1 = 0$.

Solution:

$$\begin{aligned}2(\sin^6 \theta + \cos^6 \theta) - 3(\sin^4 \theta + \cos^4 \theta) + 1 &= 2[(\sin^2 \theta)^3 + (\cos^2 \theta)^3] - 3[(\sin^2 \theta)^2 + (\cos^2 \theta)^2] + 1 \\&= 2[(\sin^2 \theta + \cos^2 \theta)^3 - 3\sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta)] - 3[(\sin^2 \theta + \cos^2 \theta)^2]\end{aligned}$$

$$- 2\sin^2\theta \cos^2\theta] + 1$$

The algebraic identity

$$a^3 + b^3 = (a + b)^3 - 3ab(a + b) \text{ and}$$

$$a^2 + b^2 = (a + b)^2 - 2ab$$

are used in the above step where

$$a = \sin^2\theta \text{ and } b = \cos^2\theta.$$

Writing $\sin^2\theta + \cos^2\theta = 1$, we have

$$= 2[1 - 3\sin^2\theta \cos^2\theta] - 3[-2\sin^2\theta \cos^2\theta] + 1$$

$$= 2 - 6\sin^2\theta \cos^2\theta - 3 + 6\sin^2\theta \cos^2\theta + 1$$

$$= -3 + 3 = 0$$

Question-16

Evaluate $\cos(40^\circ + \theta) - \sin(50^\circ - \theta) + \frac{\cos^2 40^\circ + \cos^2 50^\circ}{\sin^2 40^\circ + \sin^2 50^\circ}$.

Solution:

$$\begin{aligned} & \cos(40^\circ + \theta) - \sin(50^\circ - \theta) + \frac{\cos^2 40^\circ + \cos^2 50^\circ}{\sin^2 40^\circ + \sin^2 50^\circ} \\ &= \cos[(90^\circ - 50^\circ) + \theta] - \sin(50^\circ - \theta) + \frac{\cos^2(90^\circ - 50^\circ) + \cos^2 50^\circ}{\sin^2(90^\circ - 50^\circ) + \sin^2 50^\circ} \\ &= \sin(50^\circ - \theta) - \sin(50^\circ - \theta) + \frac{\sin^2 50^\circ + \cos^2 50^\circ}{\cos^2 50^\circ + \sin^2 50^\circ} \\ &= \frac{1}{1} \\ &= 1. \end{aligned}$$

Question-17

If $x \sin^3 \theta + y \cos^3 \theta = \sin \theta \cos \theta$ and $x \sin \theta = y \cos \theta$, prove that $x^2 + y^2 = 1$.

Solution:

$$x \sin^3 \theta + y \cos^3 \theta = \sin \theta \cos \theta$$

$$\begin{aligned} x \sin \theta (\sin^2 \theta) + (y \cos \theta) \cos^2 \theta &= \sin \theta \cos \theta \Rightarrow x \sin \theta (\sin^2 \theta) + (x \sin \theta) \cos^2 \theta = \sin \theta \cos \theta \\ & \text{[since } x \sin \theta = y \cos \theta \text{]} \Rightarrow x \sin \theta (\sin^2 \theta + \cos^2 \theta) = \sin \theta \cos \theta \\ & \Rightarrow x \sin \theta = \sin \theta \cos \theta \Rightarrow x = \cos \theta \dots x \sin \theta = y \cos \theta \Rightarrow y = \sin \theta \end{aligned}$$

$$\text{Hence } x^2 + y^2 = \cos^2 \theta + \sin^2 \theta = 1.$$

Question-18

Evaluate, $\sec^2 10^\circ - \cot^2 80^\circ + \frac{\sin 15^\circ \cos 75^\circ + \cos 15^\circ \sin 75^\circ}{\cos \theta \sin(90^\circ - \theta) + \sin \theta \cos(90^\circ - \theta)}$.

Solution:

$$\begin{aligned}\text{L.H.S.} &= \sec^2 10^\circ - \cot^2 80^\circ + \frac{\sin 15^\circ \cos 75^\circ + \cos 15^\circ \sin 75^\circ}{\cos \theta \sin(90^\circ - \theta) + \sin \theta \cos(90^\circ - \theta)} \\&= \sec^2(90^\circ - 80^\circ) - \cot^2 80^\circ + \frac{\sin(90^\circ - 75^\circ) \cos 75^\circ + \cos(90^\circ - 75^\circ) \sin 75^\circ}{\cos \theta \sin(90^\circ - \theta) + \sin \theta \cos(90^\circ - \theta)} \\&= \operatorname{cosec}^2 80^\circ - \cot^2 80^\circ + \frac{\cos 75^\circ \times \cos 75^\circ + \sin 75^\circ \times \sin 75^\circ}{\cos \theta \times \cos \theta + \sin \theta \times \sin \theta} \\&= \frac{1}{\sin^2 80^\circ} - \frac{\cos^2 80^\circ}{\sin^2 80^\circ} + \frac{1}{1} \\&= \frac{1 - \cos^2 80^\circ}{\sin^2 80^\circ} + 1 \\&= \frac{\sin^2 80^\circ}{\sin^2 80^\circ} + 1 \\&= 1 + 1 = 2 = \text{R.H.S.}\end{aligned}$$

Question-19

Prove the following identity: $\frac{\cot^2 A(\sec A - 1)}{(1 + \sin A)} + \frac{\sec^2 A(\sin A - 1)}{(1 + \sec A)} = 0$.

Solution:

$$\begin{aligned}\text{LHS} &= \frac{\cot^2 A(\sec A - 1)}{(1 + \sin A)} + \frac{\sec^2 A(\sin A - 1)}{(1 + \sec A)} \\&= \frac{\cot^2 A(\sec^2 A - 1) + \sec^2 A(\sin^2 A - 1)}{(1 + \sin A)(1 + \sec A)} \\&= \frac{\cot^2 A \tan^2 A - \sec^2 A \cos^2 A}{(1 + \sin A)(1 + \sec A)} \\&= \frac{1 - 1}{(1 + \sin A)(1 + \sec A)} = 0.\end{aligned}$$

Question-20

Prove: $\frac{\tan^2 \theta}{\tan^2 \theta - 1} + \frac{\operatorname{cosec}^2 \theta}{\sec^2 \theta - \operatorname{cosec}^2 \theta} = \frac{1}{\sin^2 \theta - \cos^2 \theta}$.

Solution:

$$\begin{aligned}\text{L.H.S.} &= \frac{\tan^2 \theta}{\tan^2 \theta - 1} + \frac{\operatorname{cosec}^2 \theta}{\sec^2 \theta - \operatorname{cosec}^2 \theta} \\&= \frac{\frac{\sin^2 \theta}{\cos^2 \theta}}{\frac{\sin^2 \theta}{\cos^2 \theta} - 1} + \frac{\frac{1}{\sin^2 \theta}}{\frac{1}{\cos^2 \theta} - \frac{1}{\sin^2 \theta}} \\&= \frac{\frac{\sin^2 \theta}{\cos^2 \theta}}{\frac{\sin^2 \theta - \cos^2 \theta}{\cos^2 \theta}} + \frac{\frac{1}{\sin^2 \theta}}{\frac{\sin^2 \theta - \cos^2 \theta}{\cos^2 \theta \sin^2 \theta}}\end{aligned}$$



$$\begin{aligned}
 &= \frac{\sin^2 \theta}{\cos^2 \theta} \times \frac{\cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} + \frac{1}{\sin^2 \theta} \times \frac{\cos^2 \theta \sin^2 \theta}{\sin^2 \theta - \cos^2 \theta} \\
 &= \frac{\sin^2 \theta}{\sin^2 \theta - \cos^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} \\
 &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} \\
 &= \frac{1}{\sin^2 \theta - \cos^2 \theta} \\
 &= \text{R.H.S.} \therefore \frac{\tan^2 \theta}{\tan^2 \theta - 1} + \frac{\operatorname{cosec}^2 \theta}{\sec^2 \theta - \operatorname{cosec}^2 \theta} = \frac{1}{\sin^2 \theta - \cos^2 \theta}
 \end{aligned}$$

Question-21

Solve the following equation for $0^\circ < \theta \leq 90^\circ$: $3 \tan \theta + \cot \theta = 5 \operatorname{cosec} \theta$.

Solution:

$$\begin{aligned}
 3 \tan \theta + \cot \theta &= 5 \operatorname{cosec} \theta \Rightarrow 3 \tan \theta + \frac{1}{\tan \theta} = 5 \operatorname{cosec} \theta \Rightarrow 3 \tan^2 \theta + 1 = 5 \operatorname{cosec} \theta \tan \theta \\
 \operatorname{cosec} \theta \tan \theta &\Rightarrow 3 \tan^2 \theta + 1 = 5 \sec \theta \Rightarrow 3(\sec^2 \theta - 1) + 1 = 5 \sec \theta \Rightarrow 3 \sec^2 \theta - 5 \sec \theta - 2 = 0 \\
 \Rightarrow \sec \theta &= \frac{5 \pm \sqrt{25 + 24}}{6} = \frac{5 \pm 7}{6} \\
 \sec \theta &= 2 \text{ or } \frac{-1}{3} \\
 \sec \theta &\neq \frac{-1}{3} \text{ (as } -1 < \cos \theta < 1) \therefore \theta = 60^\circ \text{ for } 0^\circ < \theta \leq 90^\circ.
 \end{aligned}$$

Question-22

Solve for θ : $\frac{\cos^2 \theta - 3 \cos \theta + 2}{\sin^2 \theta} = 1, (\theta \neq 0)$.

Solution:

$$\begin{aligned}
 \frac{\cos^2 \theta - 3 \cos \theta + 2}{\sin^2 \theta} &= 1 \\
 \frac{1 - \sin^2 \theta - 3 \cos \theta + 2}{\sin^2 \theta} &= 1 \\
 \frac{-\sin^2 \theta - 3 \cos \theta + 3}{\sin^2 \theta} &= 1 \\
 -\sin^2 \theta - 3 \cos \theta + 3 &= \sin^2 \theta - 2 \sin^2 \theta - 3 \cos \theta = -3 \\
 -2(\cos^2 \theta - 1) - 3 \cos \theta &= -3 \\
 -2 \cos^2 \theta + 2 - 3 \cos \theta &= -3 \\
 -2 \cos^2 \theta + 5 - 3 \cos \theta &= 0 \\
 2 \cos^2 \theta + 3 \cos \theta - 5 &= 0 \\
 \cos \theta &= \frac{-3 \pm \sqrt{9 + 40}}{4} = \frac{-3 \pm \sqrt{49}}{4} = \frac{-3 \pm 7}{4} = -\frac{5}{2} \text{ or } 1 \text{ if } \cos \theta = 1, \therefore \theta = 0^\circ.
 \end{aligned}$$

Question-23

Using the formula $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$, obtain the value of $\tan 15^\circ$.

Solution:

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Putting $\theta = 15^\circ$,

$$\tan 30^\circ = \frac{2 \tan 15^\circ}{1 - \tan^2 15^\circ}$$

$$\frac{1}{\sqrt{3}} = \frac{2 \tan 15^\circ}{1 - \tan^2 15^\circ}$$

$$(1 - \tan^2 15^\circ) = \sqrt{3}(2 \tan 15^\circ)$$

$$\tan^2 15^\circ + 2\sqrt{3} \tan 15^\circ - 1 = 0$$

$$\tan 15^\circ = \frac{-2\sqrt{3} \pm \sqrt{12 + 4}}{2} = \frac{-2\sqrt{3} \pm 4}{2} = -\sqrt{3} \pm 2.$$

Question-24

If $a \cos \theta - b \sin \theta = x$ and $a \sin \theta + b \cos \theta = y$, prove that $a^2 + b^2 = x^2 + y^2$.

Solution:

$$a \cos \theta - b \sin \theta = x \text{ and } a \sin \theta + b \cos \theta = y$$

$$\text{R.H.S.} = x^2 + y^2$$

$$= (a \cos \theta - b \sin \theta)^2 + (a \sin \theta + b \cos \theta)^2$$

$$= a^2 \cos^2 \theta - 2ab \cos \theta \sin \theta + b^2 \sin^2 \theta + a^2 \sin^2 \theta + 2ab \sin \theta \cos \theta +$$

$$b^2 \cos^2 \theta = (a^2 + b^2) \cos^2 \theta + (b^2 + a^2) \sin^2 \theta = (a^2 + b^2) \cos^2 \theta + (a^2$$

$$+ b^2) \sin^2 \theta = (a^2 + b^2) (\cos^2 \theta + \sin^2 \theta)$$

$$= (a^2 + b^2) \quad [\because \cos^2 \theta + \sin^2 \theta = 1]$$

$$= \text{L.H.S.} \therefore a^2 + b^2 = x^2 + y^2.$$

Question-25

If $x = p \sec \theta + q \tan \theta$ and $y = p \tan \theta + q \sec \theta$, prove that $x^2 - y^2 = p^2 - q^2$.

Solution:

$$x = p \sec \theta + q \tan \theta \text{ and } y = p \tan \theta + q \sec \theta$$

$$\text{L.H.S.} = x^2 - y^2$$

$$= (p \sec \theta + q \tan \theta)^2 - (p \tan \theta + q \sec \theta)^2$$

$$= p^2 \sec^2 \theta + 2pq \sec \theta \tan \theta + q^2 \tan^2 \theta - (p^2 \tan^2 \theta + 2pq \tan \theta \sec \theta + q^2 \sec^2 \theta)$$

$$= p^2 \sec^2 \theta + 2pq \sec \theta \tan \theta + q^2 \tan^2 \theta - p^2 \tan^2 \theta - 2pq \tan \theta \sec \theta - q^2$$

$$\sec^2 \theta = (p^2 - q^2) \sec^2 \theta + (q^2 - p^2) \tan^2 \theta$$

$$= (p^2 - q^2) \sec^2 \theta - (p^2 - q^2) \tan^2 \theta = (p^2 - q^2) (\sec^2 \theta - \tan^2 \theta)$$

$$= (p^2 - q^2) [\text{Since } 1 + \tan^2 \theta = \sec^2 \theta]$$

$$= \text{R.H.S.} \therefore x^2 - y^2 = p^2 - q^2.$$

Question-26

Show that $(1 + \cot \theta - \operatorname{cosec} \theta)(1 + \tan \theta + \sec \theta) = 2$.

Solution:

$$\text{L.H.S} = (1 + \cot \theta - \operatorname{cosec} \theta)(1 + \tan \theta + \sec \theta)$$

$$= \left(1 + \frac{\cos \theta}{\sin \theta} - \frac{1}{\sin \theta}\right) \left(1 + \frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta}\right)$$

$$= \left(\frac{\sin \theta + \cos \theta - 1}{\sin \theta}\right) \left(\frac{\sin \theta + \cos \theta + 1}{\cos \theta}\right)$$

$$= \frac{(\sin \theta + \cos \theta)^2 - 1}{\sin \theta \cos \theta}$$

$$= \frac{2 \sin \theta \cos \theta + \sin^2 \theta + \cos^2 \theta - 1}{\sin \theta \cos \theta}$$

$$= \frac{1 + 2 \sin \theta \cos \theta - 1}{\sin \theta \cos \theta}$$

$$= \frac{2 \sin \theta \cos \theta}{\sin \theta \cos \theta}$$

$$= 2$$

$$= \text{R.H.S.}$$

Question-27

If $\operatorname{cosec} \theta - \sin \theta = a$, $\sec \theta - \cos \theta = b$, prove that $a^2 b^2 (a^2 + b^2 + 3) = 1$.

Solution:

$$\text{L.H.S} = a^2 b^2 (a^2 + b^2 + 3)$$

$$= (\operatorname{cosec} \theta - \sin \theta)^2 (\sec \theta - \cos \theta)^2 [(\operatorname{cosec} \theta - \sin \theta)^2 + (\sec \theta - \cos \theta)^2 + 3]$$

$$= (\operatorname{cosec} \theta - \sin \theta)^2 (\sec \theta - \cos \theta)^2 [\operatorname{cosec}^2 \theta + \sin^2 \theta - 2 \operatorname{cosec} \theta \sin \theta + \sec^2 \theta + \cos^2 \theta - 2 \sec \theta \cos \theta + 3]$$

$$= (\operatorname{cosec} \theta - \sin \theta)^2 (\sec \theta - \cos \theta)^2 [\operatorname{cosec}^2 \theta + 1 - 2 + \sec^2 \theta - 2 + 3]$$

$$= (\operatorname{cosec} \theta - \sin \theta)^2 (\sec \theta - \cos \theta)^2 (\operatorname{cosec}^2 \theta + \sec^2 \theta)$$

$$= \left(\frac{1}{\sin \theta} - \sin \theta\right)^2 \left(\frac{1}{\cos \theta} - \cos \theta\right)^2 \left(\frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta}\right)$$

$$= \left(\frac{1 - \sin^2 \theta}{\sin \theta}\right)^2 \left(\frac{1 - \cos^2 \theta}{\cos \theta}\right)^2 \left(\frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta \cos^2 \theta}\right)$$

$$= \left(\frac{\cos^2 \theta}{\sin \theta}\right)^2 \left(\frac{\sin^2 \theta}{\cos \theta}\right)^2 \left(\frac{1}{\sin^2 \theta \cos^2 \theta}\right)$$

$$= \frac{\cos^4 \theta}{\sin^2 \theta} \times \frac{\sin^4 \theta}{\cos^2 \theta} \times \frac{1}{\sin^2 \theta \cos^2 \theta}$$

$$= 1$$

$$= \text{R.H.S}$$

Question-28

Prove that $(\sin \theta + \sec \theta)^2 + (\cos \theta + \operatorname{cosec} \theta)^2 = (1 + \sec \theta \operatorname{cosec} \theta)^2$.

Solution:

$$\text{L.H.S} = (\sin \theta + \sec \theta)^2 + (\cos \theta + \operatorname{cosec} \theta)^2$$

$$= \left(\sin \theta + \frac{1}{\cos \theta}\right)^2 + \left(\cos \theta + \frac{1}{\sin \theta}\right)^2$$

$$= \sin^2 \theta + \frac{1}{\cos^2 \theta} + 2 \sin \theta \frac{1}{\cos \theta} + \cos^2 \theta + \frac{1}{\sin^2 \theta} + 2 \cos \theta \frac{1}{\sin \theta}$$

$$\begin{aligned}
 &= 1 + \left(\frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} \right) + 2 \left(\sin \theta \frac{1}{\cos \theta} + \cos \theta \frac{1}{\sin \theta} \right) \\
 &= 1 + \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta \sin^2 \theta} + 2 \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \\
 &= 1 + \frac{1}{\cos^2 \theta \sin^2 \theta} + 2 \frac{1}{\cos \theta \sin \theta} \\
 &= 1 + \sec^2 \theta \operatorname{cosec}^2 \theta + 2 \sec \theta \operatorname{cosec} \theta \\
 &= (1 + \sec \theta \operatorname{cosec} \theta)^2 \\
 &= \text{R.H.S.}
 \end{aligned}$$

Question-29

Find the value of $\frac{4}{3} \tan^2 30^\circ + \sin^2 60^\circ - 3 \cos^2 60^\circ + \frac{3}{4} \tan^2 60^\circ - 2 \tan^2 45^\circ$.

Solution:

$$\begin{aligned}
 &\frac{4}{3} \tan^2 30^\circ + \sin^2 60^\circ - 3 \cos^2 60^\circ + \frac{3}{4} \tan^2 60^\circ - 2 \tan^2 45^\circ \\
 &= \frac{4}{3} \times \left(\frac{1}{\sqrt{3}} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2 - 3 \left(\frac{1}{2} \right)^2 + \frac{3}{4} (\sqrt{3})^2 - 2 \\
 &= \frac{4}{3} \times \frac{1}{3} + \frac{3}{4} - \frac{3}{4} + \frac{3}{4} \times 3 - 2 \\
 &= \frac{4}{9} + \frac{3}{4} \times 3 - 2 \\
 &= \frac{4}{9} + \frac{9}{4} - 2 \\
 &= \frac{16 + 81 - 72}{36} \\
 &= \frac{25}{36}
 \end{aligned}$$

Question-30

Find the value of $4(\sin^4 30^\circ + \cos^4 60^\circ) - 3(\cos^2 45^\circ + \sin^2 90^\circ)$.

Solution:

$$\begin{aligned}
 &4(\sin^4 30^\circ + \cos^4 60^\circ) - 3(\cos^2 45^\circ + \sin^2 90^\circ) \\
 &= 4 \left[\left(\frac{1}{2} \right)^4 + \left(\frac{1}{2} \right)^4 \right] - 3 \left[\left(\frac{1}{\sqrt{2}} \right)^2 + 1^2 \right] \\
 &= 4 \left[2 \times \frac{1}{16} \right] - 3 \left[\frac{1}{2} + 1 \right] \\
 &= \frac{1}{2} - 3 \left[\frac{3}{2} \right] \\
 &= \frac{1}{2} - \frac{9}{2} \\
 &= -\frac{8}{2} \\
 &= -4
 \end{aligned}$$

Question-31

Prove that $\left(\frac{1}{\sec^2 \theta - \cos^2 \theta} + \frac{1}{\cos^2 \theta - \sec^2 \theta} \right) \sin^2 \theta \cos^2 \theta = \frac{1 - \sin^2 \theta \cos^2 \theta}{2 + \sin^2 \theta \cos^2 \theta}$.

Solution:

$$\begin{aligned} \left(\frac{1}{\sec^2 \theta - \cos^2 \theta} + \frac{1}{\operatorname{cosec}^2 \theta - \sin^2 \theta} \right) \sin^2 \theta \cos^2 \theta &= \left(\frac{\cos^2 \theta}{1 - \cos^4 \theta} + \frac{\sin^2 \theta}{1 - \sin^4 \theta} \right) \sin^2 \theta \cos^2 \theta \\ &= \left(\frac{\cos^2 \theta - \cos^2 \theta \sin^4 \theta + \sin^2 \theta - \sin^2 \theta \cos^4 \theta}{(1 + \cos^2 \theta) \sin^2 \theta (1 + \sin^2 \theta) \cos^2 \theta} \right) \sin^2 \theta \cos^2 \theta \\ &= \frac{1 - \cos^2 \theta \sin^2 \theta (\cos^2 \theta + \sin^2 \theta)}{1 + \cos^2 \theta + \sin^2 \theta + \cos^2 \theta \sin^2 \theta} \\ &= \frac{1 - \sin^2 \theta \cos^2 \theta}{2 + \sin^2 \theta \cos^2 \theta}. \end{aligned}$$

Question-32

Find the value of $\frac{4}{3} \tan^2 30^\circ + \sin^2 60^\circ - 3 \cos^2 60^\circ + \frac{3}{4} \tan^2 60^\circ - 2 \tan^2 45^\circ$.

Solution:

$$\begin{aligned} \frac{4}{3} \tan^2 30^\circ + \sin^2 60^\circ - 3 \cos^2 60^\circ + \frac{3}{4} \tan^2 60^\circ - 2 \tan^2 45^\circ \\ &= \frac{4}{3} \left(\frac{1}{\sqrt{3}} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2 - 3 \left(\frac{1}{2} \right)^2 + \frac{3}{4} (\sqrt{3})^2 - (2 \times 1) \\ &= \left(\frac{4}{3} \times \frac{1}{3} \right) + \left(\frac{3}{4} \right) - \left(\frac{3}{4} \right) + \left(\frac{3}{4} \times 3 \right) - 2 \\ &= \frac{4}{9} + \frac{9}{4} - 2 \\ &= \frac{16 + 81 - 72}{36} \\ &= \frac{25}{36} \end{aligned}$$